MATH1520 University Mathematics for Applications — Spring 2021

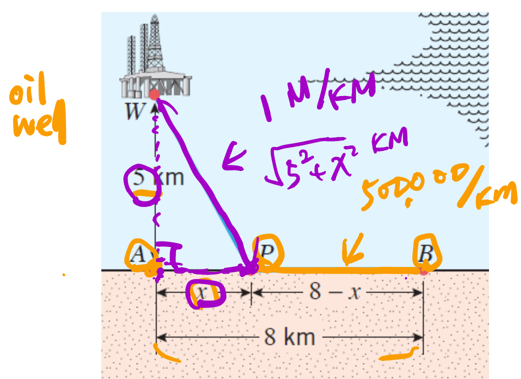
Chapter 8: Application of Derivatives III

Learning Objectives:
(1) Model and analyze optimization problems.
(2) Examine applied problems involving related rates of change.

8.1 Optimization Problem

Maximize/minimize some quantities from applied problem. This is an application of absolute extrema of functions.

Example 8.1.1. The figure shows an offshore oil well located at a point W that is 5km from the closest point A on a straight shoreline. Oil is to be piped from W to a shore point B that is 8km from A by piping it on a straight line under water from W to some shore point P between A and B and then on to B via pipe along the shoreline. If the cost of laying pipe is $\$1,000,000$/km under water and $\$500,000$/km over land, where should the point P be located to minimize the cost of laying the pipe?



Solution. Let

$$x \quad = \quad \text{distance (in kilometers) between } A \text{ and } P\text{, i.e. } |AP|$$

then,

length of pipe over land

$$|PB| = |AB| - |AP| = (8 - x) \quad \text{km}$$
$$|WP| = \sqrt{x^2 + 25} \quad \text{km}$$

when $x=\frac{5}{\sqrt{3}}$, $x^2=\frac{25}{3}$

unit: 10^6 dollars

$f(\frac{5}{\sqrt{3}}) = \sqrt{\frac{25}{3}+25} + \frac{1}{2}(8-\frac{5}{\sqrt{3}})$

$= 5\sqrt{\frac{4}{3}} + 4 - \frac{5}{2}\frac{1}{\sqrt{3}}$

$= \frac{3\cdot 5}{2\sqrt{3}} + 4 = \frac{5}{2}\sqrt{3}+4$

Then, it follows that the total cost (in million) for the pipeline is

total cost →

$$f(x) = 1(\sqrt{x^2+25}) + \frac{1}{2}(8-x) = \sqrt{x^2+25} + \frac{1}{2}(8-x), \quad x \in [0,8]. \tag{8.1}$$

f is infinitely differentiable; in particular, it is continuous on $[0,8]$. The absolute minimum exists by extreme value theorem.

find absolute min of $f(x)$, on $[0,8]$

critical pts occurs when

$$f'(x) = \underbrace{\frac{x}{\sqrt{x^2+25}}}_{\frac{1}{2}\frac{2x}{\sqrt{x^2+25}}} - \frac{1}{2} = 0$$

Setting $f'(x)=0$ and solving for x yields

$$\begin{aligned} \frac{x}{\sqrt{x^2+25}} &= \frac{1}{2} \\ x^2 &= \frac{1}{4}(x^2+25) \\ x &= \frac{5}{\sqrt{3}} \text{ or } -\frac{5}{\sqrt{3}} \quad \text{(rejected)} \end{aligned}$$

$x = \frac{\sqrt{x^2+25}}{2}$

$4x^2 - x^2 - 25 = 0$

$3x^2 = 25$

So, $x = \frac{5}{\sqrt{3}}$ is the only critical number in $(0,8)$.

Compare

left end pt [0,∞   right end pt of [0,8]

$$f(0) = 9, \quad f(\frac{5}{\sqrt{3}}) \approx 8.330, < f(8) \approx 9.433. > 9$$

crit pt.

The least possible cost of the pipeline (to the nearest dollar) is \$8,330,127, and this occurs when the point P is located at a distance of $5/\sqrt{3} \approx 2.89$ km from A.

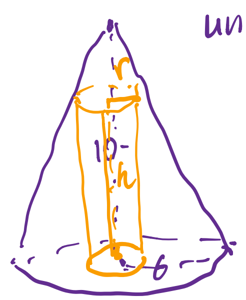
■

Procedure to solve Optimization problem:

1. Assign variables, set up a function by expressing the quantity to be optimized in terms of the independent variable.
2. Find the absolute extrema of the function.

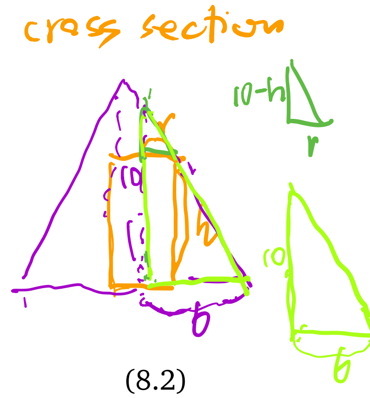
Example 8.1.2. Find the radius and height of the right circular cylinder of largest volume that can be inscribed in a right circular cone with radius 6 inches and height 10 inches.

unit: inches



Solution. Let

$$\begin{aligned} r &= \text{radius (in inches) of the cylinder} \\ h &= \text{height (in inches) of the cylinder} \\ V &= \text{volume (in cubic inches) of the cylinder} \end{aligned}$$



The formula for the volume of the inscribed cylinder is

$$V = \pi r^2 h.$$

Using similar triangles, we obtain

$6(10-h) = 10r.$

$$\frac{10-h}{r} = \frac{10}{6} \qquad \text{or} \qquad h = 10 - \frac{5}{3}r.$$

$$\therefore \qquad V(r) = \pi r^2 \left(10 - \frac{5}{3}r\right) = 10\pi r^2 - \frac{5}{3}\pi r^3 \tag{8.2}$$

which expresses V in terms of r alone. Because r represents a radius, it cannot be negative, and because the radius of the inscribed cylinder cannot exceed the radius of the cone, the variable r must satisfy

$$0 \le r \le 6 \qquad r \in [0,6]$$

Thus, we have reduced the problem to that of finding the value (or values) of r in $[0, 6]$ for which V is maximum.
From (8.2) we obtain

$$\frac{dV}{dr} = 20\pi r - 5\pi r^2 = 5\pi r(4 - r)$$

Setting $\frac{dV}{dr} = 0$ gives

critical pts
$$5\pi r(4 - r) = 0,$$

so $r = 0$ and $r = 4$ are critical points. Since these lie in the interval $[0, 6]$, the maximum must occur at one of the values

left end pt, critical pt

$$r = 0, \qquad r = 4, \qquad r = 6.$$

right end pt of $[0,6]$

Substituting these values into (8.2), we have

$$V = 0, \qquad V(4) = \frac{160}{3}\pi, \qquad V = 0$$

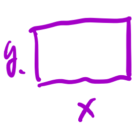
It tells us the maximum volume $V = \frac{160}{3}\pi$ occurs when the inscribed cylinder has radius 4 in. When $r = 4$ it follows that $h = \frac{10}{3}$. Thus, the inscribed cylinder of largest volume has radius $r = 4$ in and height $h = \frac{10}{3}$ in. $= 10 - \frac{5}{3}r$ ■

Example 8.1.3. Among all the rectangles with fixed area $S_0 > 0$, find the minimal perimeter.

Solution. Let one side of the rectangle has length $x > 0$ then the other side is $\frac{S_0}{x}$, and the perimeter is

$$\text{Perimeter} \quad f(x) = 2(x + \frac{S_0}{x}), \quad x \in (0, +\infty)$$

$= 2x + 2y$



Area $= S_0 = xy$

$y = \frac{S_0}{x}$

Although extreme value theorem cannot be applied on $(0, +\infty)$, we can still use the monotonicity to find the absolute extrema.

Let

$\pm 1 = \frac{\sqrt{S_0}}{x}$

$$f'(x) = 2(1 - \frac{S_0}{x^2}) = 0, \quad \Rightarrow \quad x = \sqrt{S_0} \quad \text{or} \quad -\cancel{\sqrt{S_0}} \text{ (rejected, not in } (0, +\infty))$$

critical pts

$f(\sqrt{S_0})$

x	$(0, \sqrt{S_0})$	$\sqrt{S_0}$	$(\sqrt{S_0}, +\infty)$
$f'(x)$	$-$	0	$+$
f	$\downarrow$	absolute min	$\uparrow$

Thus the minimal perimeter occurs when $x = \sqrt{S_0}$, i.e. it is a square. ■

minimal perimeter $= 2(\sqrt{S_0} + \frac{S_0}{\sqrt{S_0}})$
$= 4\sqrt{S_0}.$

8.2 Related Rates

Given rate of change of one quantity A, find the rate of change of another quantity B which is related to A. This is an application of implicit differentiation.

Example 8.2.1. A 26-foot ladder is placed against a wall. If the top of the ladder is sliding down the wall at 2 feet per second, at what rate is the bottom of the ladder moving away from the wall when the bottom of the ladder is 10 feet away from the wall?

Solution. At any time t, let

$$\begin{aligned} x(t) &= \text{the distance of the bottom of the ladder from the wall} \\ y(t) &= \text{the distance of the top of the ladder from the ground} \end{aligned}$$

x and y are related by the Pythagorean relationship:

$$x^2(t) + y^2(t) = 26^2 \tag{8.3}$$

Differentiating the above equation implicitly with respect to t, we obtain

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0. \tag{8.4}$$

The rates $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are related by equation (8.4). This is a related-rates problem.
By assumption, $\frac{dy}{dt} = -2$ (y is decreasing at a constant rate of 2 feet per second).

When $x(t) = 10$, $y(t) = \sqrt{26^2 - 10^2} = 24$ feet.

So,

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt} = \frac{-2(24)(-2)}{2(10)} = 4.8 \text{ feet per second.}$$

The bottom of the ladder is moving away from the wall at a rate of 4.8 feet per second. ■

EXPLORE!

Refer to Example 3.4.2. Because of an increase in the speed limit, the speed past the exit is now

$S_1(t) = t^3 - 10.5t^2 + 30t + 25$

Graph $S(t)$ and $S_1(t)$ using the window [0, 6]1 by [20, 60]5. At what time between 1 P.M. and 6 P.M. is the maximum speed achieved using $S_1(t)$? At what time is the minimum speed achieved?

Compute $S(t)$ for these values of t and at the endpoints $t = 1$ and $t = 6$ to get

$$S(1) = 40.5 \qquad S(2) = 46 \qquad S(5) = 32.5 \qquad S(6) = 38$$

Since the largest of these values is $S(2) = 46$ and the smallest is $S(5) = 32.5$, we can conclude that the traffic is moving fastest at 2:00 P.M., when its speed is 46 miles per hour, and slowest at 5:00 P.M., when its speed is 32.5 miles per hour. For reference, the graph of S is sketched in Figure 3.37.

EXAMPLE 3.4.3 Finding Maximum Air Speed During a Cough

When you cough, the radius of your trachea (windpipe) decreases, affecting the speed of the air in the trachea. If r_0 is the normal radius of the trachea, the relationship between the speed S of the air and the radius r of the trachea during a cough is given by a function of the form $S(r) = ar^2(r_0 - r)$, where a is a positive constant.* Find the radius r for which the speed of the air is greatest.

Solution

The radius r of the contracted trachea cannot be greater than the normal radius r_0 or less than zero. Hence, the goal is to find the absolute maximum of $S(r)$ on the interval $0 \le r \le r_0$.

First differentiate $S(r)$ with respect to r using the product rule and factor the derivative as follows (note that a and r_0 are constants):

$$S'(r) = -ar^2 + (r_0 - r)(2ar) = ar[-r + 2(r_0 - r)] = ar(2r_0 - 3r)$$

Then set the factored derivative equal to zero and solve to get the critical numbers:

$$ar(2r_0 - 3r) = 0$$

$$r = 0 \quad \text{or} \quad r = \frac{2}{3}r_0$$

critical pts of $S(r)$

Both of these values of r lie in the interval $0 \le r \le r_0$, and one is actually an endpoint of the interval. Compute $S(r)$ for these two values of r and for the other endpoint $r = r_0$ to get

left end pt

right end pt. of $[0, r_0]$

$$S(0) = 0 \qquad S\left(\frac{2}{3}r_0\right) = \frac{4a}{27}r_0^3 \qquad S(r_0) = 0$$

Compare these values and conclude that the speed of the air is greatest when the radius of the contracted trachea is $\frac{2}{3}r_0$, that is, when it is two-thirds the radius of the uncontracted trachea.

A graph of the function $S(r)$ is given in Figure 3.38. Note that the r intercepts of the graph are obvious from the factored function $S(r) = ar^2(r_0 - r)$. Notice also that the graph has a horizontal tangent when $r = 0$, reflecting the fact that $S'(0) = 0$.

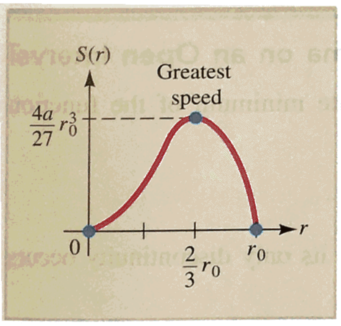


FIGURE 3.38 The speed of air during a cough $S(r) = ar^2(r_0 - r)$.

*Philip M. Tuchinsky, "The Human Cough," *UMAP Modules 1976: Tools for Teaching*, Lexington, MA: Consortium for Mathematics and Its Application, Inc., 1977.

$r \in [0, r_0]$

$S(r) = ar^2(r_0 - r) \qquad a > 0$

Find where $S(r)$ is abs. max.

Candidates:

- critical points

$$S' = a(2r)(r_0 - r) + ar^2(-1)$$

$$= a\left[2r_0 r - 3r^2\right]$$

$$= a \cdot r \cdot \left[2r_0 - 3r\right]$$

$$= 0$$

when $r = 0$ or $\frac{2r_0}{3}$

$S(0) = 0$. $\quad S\left(\frac{2r_0}{3}\right) = a\left(\frac{4r_0^2}{9}\right)\left(r_0 - \frac{2r_0}{3}\right)$

$$= a r_0^3 \left(\frac{4}{9} \cdot \frac{1}{3}\right)$$

$$= a r_0^3 \frac{4}{27} > 0$$

- end pts of $[0, r_0]$

$S(0) = 0 \qquad S(r_0) = a r_0^2 \cdot 0 = 0$

So the abs max of S occurs at

$$r = \frac{2r_0}{3}$$

a.